Chapter 9. Dynamics of Rigid Bodies

(Most of the material presented in this chapter is taken from Thornton and Marion, Chap. 11.)

9.1 Notes on Notation

In this chapter, unless otherwise stated, the following notation conventions will be used:

1. Einstein's summation convention. Whenever an index appears twice (an only twice), then a summation over this index is implied. For example,

$$x_i x_i \equiv \sum_i x_i x_i = \sum_i x_i^2. \tag{9.1}$$

2. The index i is reserved for Cartesian coordinates. For example, x_i, for $i = 1,2,3$, represents either x, y, or z depending on the value of i. Similarly, p_i can represent p_x, p_y, or p_z. This does **not** mean that any other indices cannot be used for Cartesian coordinates, but that the index i will only be used for Cartesian coordinates.
3. When dealing with systems containing multiple particles, the index α will be used to identify quantities associated with a given particle when using Cartesian coordinates. For example, if we are in the presence of n particles, the position vector for particle α is given by $\mathbf{r}_\alpha$, and its kinetic energy T_α by

$$T_\alpha = \frac{1}{2} m_\alpha \dot{x}_{\alpha,i} \dot{x}_{\alpha,i}, \qquad \alpha = 1,2, \ldots, n \text{ and } i = 1,2,3. \tag{9.2}$$

Take note that, according to convention 1 above, there is an implied summation on the Cartesian velocity components (the index i is used), but **not** on the masses since the index α appears more than twice. Correspondingly, the total kinetic energies is written as

$$T = \frac{1}{2}\sum_{\alpha=1}^{n} m_\alpha \dot{x}_{\alpha,i} \dot{x}_{\alpha,i} = \frac{1}{2}\sum_{\alpha=1}^{n} m_\alpha \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right). \tag{9.3}$$

9.2 The Independent Coordinates of a Rigid Body

The simplest extended-body model that can be treated is that of a rigid body, one in which the distances $\left|\mathbf{r}_i - \mathbf{r}_j\right|$ between points are held fixed. A general rigid body will have six degrees of freedom (but not always, see below). To see how we can specify the position of *all* points in the body with only six parameters, let us first fix some point $\mathbf{r}_1$ of the body, thereafter treated as its "centre" or origin from which all other points in the body can be referenced from ($\mathbf{r}_1$ can be, but not necessarily, the centre of mass). Once the coordinates of $\mathbf{r}_1$ are specified (in relation to some origin outside of the body), we have already used up three degrees of freedom. With $\mathbf{r}_1$ fixed, the position of some other point $\mathbf{r}_2$ can be specified using only two coordinates since it is constrained to move on the surface of a sphere centered on $\mathbf{r}_1$. We are now up to five degrees of freedom. If we now consider any other third point $\mathbf{r}_3$ not located on the axis joining $\mathbf{r}_1$ and $\mathbf{r}_2$, its position can be specified using one degree of freedom (or coordinate) for it can only rotate about the axis connecting $\mathbf{r}_1$ and $\mathbf{r}_2$. We thus have used up the six degrees of freedom. It is interesting to note that in the case of a linear rod, any point $\mathbf{r}_3$ must lay on the axis joining $\mathbf{r}_1$ and $\mathbf{r}_2$; hence a linear rod has only five degrees of freedom.

Usually, the six degrees of freedom are divided in two groups: three degrees for translation (to specify the position of the "centre" $\mathbf{r}_1$, and three rotation angles to specify the orientation of the rigid body (normally taken to be the so-called **Euler angles**).

9.3 The Inertia Tensor

Let's consider a rigid body composed of n particles of mass m_α, $\alpha = 1, \ldots, n$. If the body rotates with an angular velocity $\boldsymbol{\omega}$ about some point fixed with respect to the body coordinates (this "body" coordinate system is what we used to refer to as "noninertial" or "rotating" coordinate system in Chapter 8), and if this point moves linearly with a velocity $\mathbf{V}$ with respect to a fixed (i.e., inertial) coordinate system, then the velocity of the αth particle is given by equation (8.16) of derived in Chapter 8

$$\mathbf{v}_\alpha = \mathbf{V} + \boldsymbol{\omega} \times \mathbf{r}_\alpha, \tag{9.4}$$

where we omitted the term

$$\mathbf{v}_{\alpha,r} \equiv \left(\frac{d\mathbf{r}_\alpha}{dt} \right)_{\text{rotating}} = 0, \tag{9.5}$$

since we are dealing with a rigid body. We have also dropped the f subscript, denoting the fixed coordinate system, as it is understood that all the non-vanishing velocities will be measured in this system; again, we are dealing with a rigid body.

The total kinetic energy of the body is given by

$$\begin{aligned} T &= \sum_\alpha T_\alpha = \frac{1}{2} \sum_\alpha m_\alpha v_\alpha{}^2 \\ &= \frac{1}{2} \sum_\alpha m_\alpha \left(\mathbf{V} + \boldsymbol{\omega} \times \mathbf{r}_\alpha \right)^2 \\ &= \frac{1}{2} \sum_\alpha m_\alpha V^2 + \sum_\alpha m_\alpha \mathbf{V} \cdot \left(\boldsymbol{\omega} \times \mathbf{r}_\alpha \right) + \frac{1}{2} \sum_\alpha m_\alpha \left(\boldsymbol{\omega} \times \mathbf{r}_\alpha \right)^2. \end{aligned} \tag{9.6}$$

Although this is an equation for the total kinetic energy is perfectly general, considerable simplification will result if we choose the origin of the body coordinate system to coincide with the centre of mass. With this choice, the second term on the right hand side of the last of equations (9.6) can be seen to vanish from

$$\sum_\alpha m_\alpha \mathbf{V}\cdot(\boldsymbol{\omega}\times\mathbf{r}_\alpha) = \mathbf{V}\cdot\left[\boldsymbol{\omega}\times\left(\sum_\alpha m_\alpha \mathbf{r}_\alpha\right)\right] = 0, \tag{9.7}$$

since the centre of mass $\mathbf{R}$ of the body, of mass M, is defined such that

$$\sum_\alpha m_\alpha \mathbf{r}_\alpha = 0. \tag{9.8}$$

The total kinetic energy can then be broken into two components: one for the translational kinetic energy and another for the rotational kinetic energy. That is,

$$T = T_{\text{trans}} + T_{\text{rot}}, \tag{9.9}$$

with

$$\boxed{\begin{aligned} T_{\text{trans}} &= \frac{1}{2}\sum_\alpha m_\alpha V^2 = \frac{1}{2}MV^2 \\ T_{\text{rot}} &= \frac{1}{2}\sum_\alpha m_\alpha(\boldsymbol{\omega}\times\mathbf{r}_\alpha)^2 \end{aligned}} \tag{9.10}$$

The expression for T_{rot} can be further modified, but to do so we will now resort to tensor (or index) notation. So, let's consider the following vector equation

$$(\boldsymbol{\omega}\times\mathbf{r}_\alpha)^2 = (\boldsymbol{\omega}\times\mathbf{r}_\alpha)\cdot(\boldsymbol{\omega}\times\mathbf{r}_\alpha), \tag{9.11}$$

and rewrite it using the Levi-Civita and the Kronecker tensors

$$\begin{aligned}(\varepsilon_{ijk}\omega_j x_{\alpha,k})(\varepsilon_{imn}\omega_m x_{\alpha,n}) &= \varepsilon_{ijk}\varepsilon_{imn}\omega_j x_{\alpha,k}\omega_m x_{\alpha,n} \\ &= (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km})\omega_j x_{\alpha,k}\omega_m x_{\alpha,n} \\ &= \omega_j\omega_j x_{\alpha,k}x_{\alpha,k} - \omega_j x_{\alpha,j}\omega_k x_{\alpha,k}.\end{aligned} \tag{9.12}$$

Inserting this result in the equation for T_{rot} in equation (9.10) we get

$$T_{\text{rot}} = \frac{1}{2}\sum_\alpha m_\alpha\left[|\boldsymbol{\omega}|^2|\mathbf{r}_\alpha|^2 - (\boldsymbol{\omega}\cdot\mathbf{r}_\alpha)^2\right]. \tag{9.13}$$

Alternatively, keeping with the tensor notation we have

$$
\begin{aligned}
T_{\text{rot}} &= \frac{1}{2}\sum_{\alpha} m_\alpha \left[\omega_j \omega_j x_{\alpha,k} x_{\alpha,k} - \omega_i x_{\alpha,i} \omega_j x_{\alpha,j} \right] \\
&= \frac{1}{2}\sum_{\alpha} m_\alpha \left[\left(\omega_i \omega_j \delta_{ij} \right) x_{\alpha,k} x_{\alpha,k} - \omega_i x_{\alpha,i} \omega_j x_{\alpha,j} \right] \\
&= \frac{1}{2}\left(\omega_i \omega_j \right) \sum_{\alpha} m_\alpha \left[\delta_{ij} x_{\alpha,k} x_{\alpha,k} - x_{\alpha,i} x_{\alpha,j} \right].
\end{aligned}
\tag{9.14}
$$

We now define the components I_{ij} of the so-called inertia tensor $\{\mathbf{I}\}$ by

$$
\boxed{I_{ij} = \sum_{\alpha} m_\alpha \left[\delta_{ij} x_{\alpha,k} x_{\alpha,k} - x_{\alpha,i} x_{\alpha,j} \right]}
\tag{9.15}
$$

and the rotational kinetic energy becomes

$$
\boxed{T_{\text{rot}} = \frac{1}{2} I_{ij} \omega_i \omega_j}
\tag{9.16}
$$

or in vector notation

$$
\boxed{T_{\text{rot}} = \frac{1}{2} \boldsymbol{\omega} \cdot \{\mathbf{I}\} \cdot \boldsymbol{\omega}}
\tag{9.17}
$$

For our purposes it will be usually sufficient to treat the inertia tensor as a regular 3×3 matrix. Indeed, we can explicitly write $\{\mathbf{I}\}$ using equation (9.15) as

$$
\{\mathbf{I}\} = \left\{
\begin{array}{ccc}
\sum_{\alpha} m_\alpha \left(x_{\alpha,2}{}^2 + x_{\alpha,3}{}^2 \right) & -\sum_{\alpha} m_\alpha x_{\alpha,1} x_{\alpha,2} & -\sum_{\alpha} m_\alpha x_{\alpha,1} x_{\alpha,3} \\
-\sum_{\alpha} m_\alpha x_{\alpha,2} x_{\alpha,1} & \sum_{\alpha} m_\alpha \left(x_{\alpha,1}{}^2 + x_{\alpha,3}{}^2 \right) & -\sum_{\alpha} m_\alpha x_{\alpha,2} x_{\alpha,3} \\
-\sum_{\alpha} m_\alpha x_{\alpha,3} x_{\alpha,1} & -\sum_{\alpha} m_\alpha x_{\alpha,3} x_{\alpha,2} & \sum_{\alpha} m_\alpha \left(x_{\alpha,1}{}^2 + x_{\alpha,2}{}^2 \right)
\end{array}
\right\}
\tag{9.18}
$$

It is easy to see from either equation (9.15) or equation (9.18) that the inertia tensor is symmetric, that is,

$$
I_{ij} = I_{ji}.
\tag{9.19}
$$

The diagonal elements I_{11}, I_{22}, and I_{33} are called the **moments of inertia** about the x_1-, x_2-, and x_3-axes, respectively. The negatives of the off-diagonal elements are the

products of inertia. Finally, in most cases the rigid body is continuous and not made up of discrete particles as was assumed so far, but the results are easily generalized by replacing the summation by a corresponding integral in the expression for the components of the inertia tensor

$$I_{ij} = \int_V \rho(\mathbf{r})\left(\delta_{ij}x_k x_k - x_i x_j\right) dx_1 dx_2 dx_3, \tag{9.20}$$

where $\rho(\mathbf{r})$ is the mass density at the position $\mathbf{r}$, and the integral is to be performed over the whole volume V of the rigid body.

Example

Calculate the inertia tensor for a homogeneous cube of density ρ, mass M, and side length b. Let one corner be at the origin, and three adjacent edges lie along the coordinate axes (see Figure 9-1).
Solution.
We use equation (9.20) to calculate the components of the inertia tensor. Because of the symmetry of the problem, it is easy to see that the three moments of inertia I_{11}, I_{22}, and I_{33} are equal and that same holds for all of the products of inertia. So,

$$\begin{aligned} I_{11} &= \int_0^b \int_0^b \int_0^b \rho\left(x_2^{\,2} + x_3^{\,2}\right) dx_1 dx_2 dx_3 \\ &= \rho \int_0^b dx_3 \int_0^b dx_2 \left(x_2^{\,2} + x_3^{\,2}\right) \int_0^b dx_1 \\ &= \rho b \int_0^b dx_3 \left(\frac{b^3}{3} + b x_3^{\,2}\right) = \rho b \left(\frac{b^4}{3} + \frac{b^4}{3}\right) \\ &= \frac{2}{3}\rho b^5 = \frac{2}{3} M b^2. \end{aligned} \tag{9.21}$$

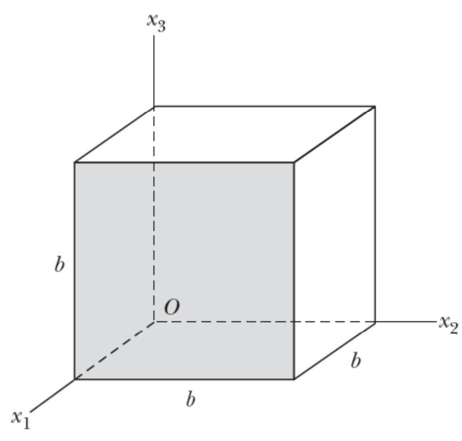

Figure 9-1 – A homogeneous cube of sides b with the origin at one corner.

And for the negative of the products of inertia

$$
\begin{aligned}
I_{12} &= -\int_0^b \int_0^b \int_0^b \rho x_1 x_2 \, dx_1 dx_2 dx_3 \\
&= -\rho \left(\frac{b^2}{2} \right) \left(\frac{b^2}{2} \right) (b) \\
&= -\frac{1}{4} \rho b^5 = -\frac{1}{4} M b^2 .
\end{aligned}
\tag{9.22}
$$

It should be noted that in this example the origin of the coordinate system is not located at the centre of mass of the cube.

9.4 Angular Momentum

Going back to the case of a rigid body composed of a discrete number of particles; we can calculate the angular momentum with respect to some point O fixed in the body coordinate system with

$$
\mathbf{L} = \sum_{\alpha} \mathbf{r}_{\alpha} \times \mathbf{p}_{\alpha} . \tag{9.23}
$$

Relative to the body coordinate system the linear momentum of the αth particle is

$$
\mathbf{p}_{\alpha} = m_{\alpha} \mathbf{v}_{\alpha} = m_{\alpha} \boldsymbol{\omega} \times \mathbf{r}_{\alpha} , \tag{9.24}
$$

and the total angular momentum becomes

$$
\mathbf{L} = \sum_{\alpha} m_{\alpha} \mathbf{r}_{\alpha} \times \left(\boldsymbol{\omega} \times \mathbf{r}_{\alpha} \right). \tag{9.25}
$$

Resorting one more time to tensor notation we can calculate $\mathbf{r}_{\alpha} \times \left(\boldsymbol{\omega} \times \mathbf{r}_{\alpha} \right)$ as

$$
\begin{aligned}
\varepsilon_{ijk} x_{\alpha,j} \varepsilon_{klm} \omega_l x_{\alpha,m} &= \varepsilon_{kij} \varepsilon_{klm} x_{\alpha,j} \omega_l x_{\alpha,m} \\
&= \left(\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl} \right) x_{\alpha,j} \omega_l x_{\alpha,m} \\
&= x_{\alpha,j} x_{\alpha,j} \omega_i - x_{\alpha,j} \omega_j x_{\alpha,i} ,
\end{aligned}
\tag{9.26}
$$

or alternatively in vector notation

$$
\mathbf{r}_{\alpha} \times \left(\boldsymbol{\omega} \times \mathbf{r}_{\alpha} \right) = r_{\alpha}^{2} \boldsymbol{\omega} - \mathbf{r}_{\alpha} \left(\mathbf{r}_{\alpha} \cdot \boldsymbol{\omega} \right). \tag{9.27}
$$

Then, the total angular momentum is given by

$$\boxed{\mathbf{L} = \sum_\alpha m_\alpha \left[r_\alpha{}^2 \boldsymbol{\omega} - \mathbf{r}_\alpha \left(\mathbf{r}_\alpha \cdot \boldsymbol{\omega} \right) \right]} \tag{9.28}$$

Using the tensor notation the component of the angular momentum is

$$\begin{aligned} L_i &= \sum_\alpha m_\alpha \left(x_{\alpha,k} x_{\alpha,k} \omega_i - x_{\alpha,j} \omega_j x_{\alpha,i} \right) \\ &= \omega_j \sum_\alpha m_\alpha \left(\delta_{ij} x_{\alpha,k} x_{\alpha,k} - x_{\alpha,j} x_{\alpha,i} \right), \end{aligned} \tag{9.29}$$

and upon using equation (9.15) for the inertia tensor

$$\boxed{L_i = I_{ij} \omega_j} \tag{9.30}$$

or in tensor notation

$$\boxed{\mathbf{L} = \{\mathbf{I}\} \cdot \boldsymbol{\omega}} \tag{9.31}$$

Finally, we can insert equation (9.31) for the angular momentum vector into equation (9.17) for the rotational kinetic energy to obtain

$$\boxed{T_{\mathrm{rot}} = \frac{1}{2} \mathbf{L} \cdot \boldsymbol{\omega}} \tag{9.32}$$

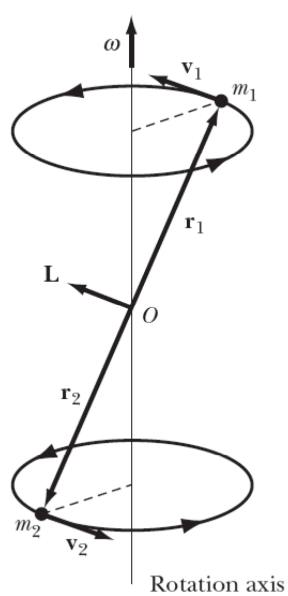

Figure 9-2 – A dumbbell connected by masses m_1 and m_2 at the ends of its shaft. Note that the angular velocity $\boldsymbol{\omega}$ is not directed along the shaft.

Example

The dumbbell. A dumbbell is connected by two masses m_1 and m_2 located at distances r_1 and r_2 from the middle of the shaft, respectively. The shaft makes an angle θ with a vertical axis, to which it is attached at its middle (i.e., the middle of the shaft). Calculate the equation of angular motion if the system is forced to rotate about the vertical axis with a constant angular velocity $\boldsymbol{\omega}$ (see Figure 9-2).

Solution. We define the inertial and the body coordinate systems such that their respective origins are both connected at the point of junction between the vertical axis and the shaft of the dumbbell. We further define the body coordinate system as having its x_3-axis orientated along the shaft and its x_1-axis perpendicular to the shaft but located in the plane defined by the axis of rotation and the shaft. The remaining x_2-axis is perpendicular to this plane and completes the coordinate system attached to the rigid body. For the inertial system, we chose the x_3'-axis to be the vertical, and the other two axes such that the basis vectors can expressed as

$$\begin{aligned}
\mathbf{e}_1' &= \cos(\theta)\cos(\omega t)\mathbf{e}_1 - \sin(\omega t)\mathbf{e}_2 - \sin(\theta)\cos(\omega t)\mathbf{e}_3 \\
\mathbf{e}_2' &= \cos(\theta)\sin(\omega t)\mathbf{e}_1 + \cos(\omega t)\mathbf{e}_2 - \sin(\theta)\sin(\omega t)\mathbf{e}_3 \\
\mathbf{e}_3' &= \sin(\theta)\mathbf{e}_1 + \cos(\theta)\mathbf{e}_3.
\end{aligned} \tag{9.33}$$

From equation (9.15), we can evaluate the components of the inertia tensor. We can in the first time identify the components that are zero (because $x_{1,1} = x_{1,2} = x_{2,1} = x_{2,2} = 0$)

$$I_{12} = I_{21} = I_{13} = I_{31} = I_{23} = I_{32} = I_{33} = 0. \tag{9.34}$$

The only two remaining components are

$$\begin{aligned}
I_{11} = I_{22} &= m_1 x_{1,3}^{\ 2} + m_2 x_{2,3}^{\ 2} \\
&= m_1 r_1^2 + m_2 r_2^2.
\end{aligned} \tag{9.35}$$

The components of the angular velocity in the coordinate of the rigid body are

$$\begin{aligned}
\omega_1 &= \omega\sin(\theta) \\
\omega_2 &= 0 \\
\omega_3 &= \omega\cos(\theta).
\end{aligned} \tag{9.36}$$

Inserting equations (9.35) and (9.36) in equation (9.30) we find for the L_1 component

$$L_1 = I_{1i}\omega_i = I_{11}\omega_1 = \omega\sin(\theta)\left(m_1 r_1^2 + m_2 r_2^2\right), \tag{9.37}$$

and for the L_2 and L_3

$$\begin{aligned} L_2 &= I_{2i}\omega_i = I_{22}\omega_2 = 0 \\ L_3 &= I_{3i}\omega_i = I_{33}\omega_3 = 0. \end{aligned} \tag{9.38}$$

If should be noted from equations (9.36) and (9.37) that the angular velocity and the angular momentum do not point in the same direction. To calculate the equations of motion, we express the angular momentum with the inertial coordinates instead of the coordinates of the rigid body system. From equation (9.37) we can write

$$\mathbf{L} = \omega \sin(\theta)\left(m_1 r_1^{\,2} + m_2 r_2^{\,2}\right)\mathbf{e}_1 = I_{11}\omega \sin(\theta)\mathbf{e}_1, \tag{9.39}$$

but we can transform the basis vector $\mathbf{e}_1$ using equation (9.33) (or its inverse)

$$\mathbf{L} = I_{11}\omega \sin(\theta)\left[\cos(\theta)\cos(\omega t)\mathbf{e}'_1 + \cos(\theta)\sin(\omega t)\mathbf{e}'_2 + \sin(\theta)\mathbf{e}'_3\right]. \tag{9.40}$$

We also know that

$$\frac{d\mathbf{L}}{dt} = \mathbf{N}, \tag{9.41}$$

where $\mathbf{N}$ is the torque. Assuming that the angular speed is constant, we find

$$\begin{aligned} N_1 &= -I_{11}\omega^2 \sin(\theta)\cos(\theta)\sin(\omega t) \\ N_2 &= I_{11}\omega^2 \sin(\theta)\cos(\theta)\cos(\omega t) \\ N_3 &= 0. \end{aligned} \tag{9.42}$$

An interesting consequence of the fact that the angular momentum and angular velocity vectors are not aligned with each other is that we need to apply a torque to the dumbbell to keep it rotating at a constant angular velocity.

9.5 The Principal Axes of Inertia

We now set on finding a set of body axes that will render the inertia tensor diagonal in form. That is, given equation (9.18) for $\{\mathbf{I}\}$, we want to make a change in the body basis vectors (i.e., a change of variables) that will change the form of the inertia tensor to

$$\{\mathbf{I}\} = \begin{Bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{Bmatrix}. \tag{9.43}$$

We will then require that all the products of inertia be zero. Carrying this program will provide a significant simplification for the expressions of the angular momentum and the kinetic energy, as measured in the inertial reference frame. That is, these two quantities will be given by

$$L_1 = I_1\omega_1, \quad L_2 = I_2\omega_2, \quad L_2 = I_3\omega_3, \tag{9.44}$$

and

$$T_{\text{rot}} = \frac{1}{2}\left(I_1\omega_1^{\;2} + I_2\omega_2^{\;2} + I_3\omega_3^{\;2}\right). \tag{9.45}$$

The set of axes that allow this transformation is called the **principal axes of inertia**. When the equations for the components of the angular momentum can be expressed as in equation (9.43), then $\mathbf{L}$, and $\boldsymbol{\omega}$ are directed along the same axis.

The problem of finding the principal axes is mathematically equivalent to solving a set of linear equations. More precisely, we have from equation (9.31) that

$$\mathbf{L} = \{\mathbf{I}\} \cdot \boldsymbol{\omega}, \tag{9.46}$$

but we are actually looking for a way to reduce this equation to the following form

$$\mathbf{L} = \{\mathbf{I}\} \cdot \boldsymbol{\omega} = I\boldsymbol{\omega}. \tag{9.47}$$

Mathematically speaking, I, which is called a **principal moment of inertia**, is an eigenvalue of the inertia tensor, and $\boldsymbol{\omega}$, which will give us the corresponding principal axis of inertia, is an eigenvector. The system of equations (9.47) can be written as

$$\begin{aligned} L_1 &= I\omega_1 = I_{11}\omega_1 + I_{12}\omega_2 + I_{13}\omega_3 \\ L_2 &= I\omega_2 = I_{21}\omega_1 + I_{22}\omega_2 + I_{23}\omega_3 \\ L_3 &= I\omega_3 = I_{31}\omega_1 + I_{32}\omega_2 + I_{33}\omega_3, \end{aligned} \tag{9.48}$$

or, after some rearranging

$$\begin{aligned} &\left(I_{11} - I\right)\omega_1 + I_{12}\omega_2 + I_{13}\omega_3 = 0 \\ &I_{21}\omega_1 + \left(I_{22} - I\right)\omega_2 + I_{23}\omega_3 = 0 \\ &I_{31}\omega_1 + I_{32}\omega_2 + \left(I_{33} - I\right)\omega_3 = 0. \end{aligned} \tag{9.49}$$

The mathematical condition necessary for this set of equation to have a nontrivial solution is that the determinant of the coefficient vanishes

$$\begin{vmatrix} (I_{11}-I)+I_{12}+I_{13} \\ I_{21}+(I_{22}-I)+I_{23} \\ I_{31}+I_{32}+(I_{33}-I) \end{vmatrix} = 0. \qquad (9.50)$$

The expansion of this determinant leads to the so-called **secular** or **characteristic equation** for the eigenvalues I (i.e., $I_1, I_2,$ and I_3 in equation (9.44)); it is a third order polynomial. Once the characteristic equation has been solved, the principal axes can be determined by inserting the eigenvalues back in equation (9.49) and evaluating the ratios of the angular velocity components ($\omega_1 : \omega_2 : \omega_3$), therefore, determining the corresponding eigenvectors.

It is important to realize that in many cases, the rigid body under study will exhibit some symmetry that will allow one to guess what the principal axes are. For example, a cylinder will have one of its principal axes directed along the centre axis of the cylinder. The two remaining axes will be directed at right angle to this axis (and to each other).

Finally, here are a few definitions: a body that has *i)* $I_1 = I_2 = I_3$ is called a **spherical top**, *ii)* $I_1 = I_2 \neq I_3$ is a **symmetric top**, *iii)* $I_1 \neq I_2 \neq I_3$ is an **asymmetric top**, and finally, if $I_1 = 0,\ I_2 = I_3$ the body is a **rotor**.

Example

Find the principal moment of inertia and the principal axes of inertia for the cube of Figure 9-1.

Solution.
In the previous example solved on page 163, we found that the products of inertia did not equal zero. Obviously, the axes chosen were not the principal axes. To find the principal moments of inertia, we must solve the characteristic equation

$$\begin{vmatrix} \frac{2}{3}\beta - I & -\frac{1}{4}\beta & -\frac{1}{4}\beta \\ -\frac{1}{4}\beta & \frac{2}{3}\beta - I & -\frac{1}{4}\beta \\ -\frac{1}{4}\beta & -\frac{1}{4}\beta & \frac{2}{3}\beta - I \end{vmatrix} = 0, \qquad (9.51)$$

where $\beta \equiv Mb^2$. Before trying to evaluate the determinant from equation (9.51), it is always good to see if we can bring it to a simpler form with a subtraction of a column or a row to another column or row. This is permitted since the determinant is not affected by such operations. In our case, we see that, for example, the third element of the first two rows is the same; we will therefore subtract the second row to the first. The determinant then becomes

$$\begin{vmatrix} \frac{11}{12}\beta - I & -\frac{11}{12}\beta + I & 0 \\ -\frac{1}{4}\beta & \frac{2}{3}\beta - I & -\frac{1}{4}\beta \\ -\frac{1}{4}\beta & -\frac{1}{4}\beta & \frac{2}{3}\beta - I \end{vmatrix} = 0, \tag{9.52}$$

which implies that

$$\left(\frac{11}{12}\beta - I\right)\begin{vmatrix} 1 & -1 & 0 \\ -\frac{1}{4}\beta & \frac{2}{3}\beta - I & -\frac{1}{4}\beta \\ -\frac{1}{4}\beta & -\frac{1}{4}\beta & \frac{2}{3}\beta - I \end{vmatrix} = 0. \tag{9.53}$$

Expanding this equation gives

$$\begin{aligned} \left(\frac{11}{12}\beta - I\right)\left[\left(\frac{2}{3}\beta - I\right)^2 - \left(-\frac{1}{4}\beta\right)^2 + \left(-\frac{1}{4}\beta\right)\left(\frac{2}{3}\beta - I\right) - \left(-\frac{1}{4}\beta\right)^2\right] &= 0 \\ \left(\frac{11}{12}\beta - I\right)\left[\left(\frac{2}{3}\beta - I\right)^2 - \left(\frac{1}{4}\beta\right)\left(\frac{2}{3}\beta - I\right) - 2\left(\frac{1}{4}\beta\right)^2\right] &= 0. \end{aligned} \tag{9.54}$$

The square-bracketed part of the last expression is a second order polynomial in $(2\beta/3 - I)$. If we solve for the roots of this polynomial we get

$$\begin{aligned} \left(\frac{2}{3}\beta - I\right) &= \frac{1}{2}\left(\frac{1}{4}\beta \pm \sqrt{\left(\frac{1}{4}\beta\right)^2 + 8\left(\frac{1}{4}\beta\right)^2}\right) \\ &= \frac{1}{8}\beta(1 \pm 3). \end{aligned} \tag{9.55}$$

The last of equation (9.54) can now be factored as

$$\left(\frac{11}{12}\beta - I\right)\left(\frac{11}{12}\beta - I\right)\left(\frac{1}{6}\beta - I\right) = 0, \tag{9.56}$$

and the three principal moments of inertia are

$$I_1 = \frac{1}{6}\beta, \quad I_2 = I_3 = \frac{11}{12}\beta. \tag{9.57}$$

To find the direction of the principal axis of inertia, we insert the eigenvalues of equation (9.57) into equation (9.49). For I_1, we have (after some manipulation)

$$\begin{aligned} 2\omega_1 - \omega_2 - \omega_3 &= 0 \\ -\omega_1 + 2\omega_2 - \omega_3 &= 0, \end{aligned} \tag{9.58}$$

which implies that $\omega_1 = \omega_2 = \omega_3$ and the corresponding eigenvector is directed along $\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$. For I_2 and I_3, because they are equal, the orientation of their corresponding principal axes is arbitrary; they need only to lie in a plane perpendicular to the main principal axis determined for I_1.

9.6 Similarity Transformations

The diagonalization of the inertia tensor (or any other matrix for that matter) discussed in the previous section can sometimes be achieved in a different manner. For example, in the case of the cube discussed in the last example, because of the symmetry of the problem we could have guessed what the principal axes were. When this can be done, it is then straightforward (if sometimes tedious) to determine the transformation (rotation) matrix that brings us from the initial set of coordinate axes to the principal axes, and use it to render the inertia tensor diagonal. In a way, this technique follows a path that is reversed from what was done in the previous section. That is, instead of, first, rendering the inertia tensor diagonal and then, determining the principal axes (i.e., the eigenvectors), we now guess the orientation of the principal axes and then diagonalize the inertia tensor.

So, let's consider the angular momentum

$$\mathbf{L} = \{\mathbf{I}\} \cdot \boldsymbol{\omega}, \tag{9.59}$$

in the initial set of coordinate axes, which we now transform to a new set of axes. We will have a new equation for the angular momentum as measured in this new system; we define the angular momentum with an equation similar to equation (9.59)

$$\mathbf{L}' = \{\mathbf{I}'\} \cdot \boldsymbol{\omega}'. \tag{9.60}$$

If we denote the transformation matrix that links the two coordinate systems by $\boldsymbol{\lambda}$ such that

$$\mathbf{L}' = \boldsymbol{\lambda}\mathbf{L}, \tag{9.61}$$

and

$$\boldsymbol{\omega}' = \boldsymbol{\lambda}\boldsymbol{\omega}. \tag{9.62}$$

We have relations similar to equations (9.61) and (9.62) for any corresponding vectors between the two systems of coordinates. For example, the position vectors are related through

$$\mathbf{r}' = \boldsymbol{\lambda}\mathbf{r}. \tag{9.63}$$

Combining equations (9.60) to (9.62) we can write

$$\mathbf{L}' = \boldsymbol{\lambda}\mathbf{L} = \{\mathbf{I}'\}\cdot\boldsymbol{\omega}' = \{\mathbf{I}'\}\cdot(\boldsymbol{\lambda}\boldsymbol{\omega}), \tag{9.64}$$

or

$$\mathbf{L} = \left(\boldsymbol{\lambda}^{-1}\{\mathbf{I}'\}\boldsymbol{\lambda}\right)\cdot\boldsymbol{\omega}. \tag{9.65}$$

Comparison with equation (9.59) reveals that

$$\boxed{\{\mathbf{I}'\} = \boldsymbol{\lambda}\{\mathbf{I}\}\boldsymbol{\lambda}^{-1}} \tag{9.66}$$

A transformation of this type is called a **similarity transformation**. In instances where we deal with orthogonal transformations (which will often be our case), we have $\boldsymbol{\lambda}^{-1} = \boldsymbol{\lambda}^T$, with $\boldsymbol{\lambda}^T$ the transpose of $\boldsymbol{\lambda}$, and

$$\{\mathbf{I}'\} = \boldsymbol{\lambda}\{\mathbf{I}\}\boldsymbol{\lambda}^T. \tag{9.67}$$

Finally, if the transformation matrix is such that it takes the initial coordinate axes into the principal axes, then, the transformed inertia tensor will be diagonal.

Example

Use the results of the preceding example of the cube (equations (9.58) and the following paragraph) to render its inertia tensor (using equations (9.21) and (9.22)) diagonal.

Solution.
From equations (9.21) and (9.22), and the corresponding set of coordinate axes defined in Figure 9-1, we can write the inertia tensor of the cube as

$$\{\mathbf{I}\} = \begin{Bmatrix} \frac{2}{3}\beta & -\frac{1}{4}\beta & -\frac{1}{4}\beta \\ -\frac{1}{4}\beta & \frac{2}{3}\beta & -\frac{1}{4}\beta \\ -\frac{1}{4}\beta & -\frac{1}{4}\beta & \frac{2}{3}\beta \end{Bmatrix}, \tag{9.68}$$

where $\beta \equiv Mb^2$. If the three initial basis vectors are denoted by $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$, we know from previous results that the main principal axis can be chosen to be

$$\mathbf{e}_1' = \frac{1}{\sqrt{3}}\left(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3\right). \tag{9.69}$$

This is equivalent to bringing the x_1-axis along the main diagonal of the cube with a $45°$ rotation. More precisely, to achieve this we first make rotation a $\boldsymbol{\lambda}_1$ of $45°$ about the x_3-axis, and a second rotation $\boldsymbol{\lambda}_2$ of $-\tan^{-1}\left(1/\sqrt{2}\right) \simeq -35°$ about the new x_2'-axis resulting from the initial rotation. The two remaining vectors necessary to complete the new basis are then

$$\begin{aligned} \mathbf{e}_2' &= \frac{1}{\sqrt{2}}\left(-\mathbf{e}_1 + \mathbf{e}_2\right) \\ \mathbf{e}_3' &= \frac{1}{\sqrt{6}}\left(-\mathbf{e}_1 - \mathbf{e}_2 + 2\mathbf{e}_3\right), \end{aligned} \tag{9.70}$$

and are seen to satisfy the following (required) relations

$$\mathbf{e}_1' \cdot \mathbf{e}_2' = \mathbf{e}_2' \cdot \mathbf{e}_3' = \mathbf{e}_2' \cdot \mathbf{e}_3' = 0. \tag{9.71}$$

From equations (9.69) and (9.70), the total transformation matrix can be written as

$$\boldsymbol{\lambda} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ -\sqrt{\dfrac{3}{2}} & \sqrt{\dfrac{3}{2}} & 0 \\ -\dfrac{1}{\sqrt{2}} & -\dfrac{1}{\sqrt{2}} & \sqrt{2} \end{bmatrix}. \tag{9.72}$$

It can be verified that $\boldsymbol{\lambda} = \boldsymbol{\lambda}_2\boldsymbol{\lambda}_1$ with

$$\boldsymbol{\lambda}_1 = \begin{bmatrix} \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} & 0 \\ -\dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{\lambda}_2 = \begin{bmatrix} \sqrt{\dfrac{2}{3}} & 0 & \dfrac{1}{\sqrt{3}} \\ 0 & 1 & 0 \\ -\dfrac{1}{\sqrt{3}} & 0 & \sqrt{\dfrac{2}{3}} \end{bmatrix}. \tag{9.73}$$

Using equation (9.67) with equations (9.68) and (9.72), we find

$$\{\mathbf{I}'\} = \boldsymbol{\lambda}\{\mathbf{I}\}\boldsymbol{\lambda}^T =$$

$$= \frac{\beta}{3}\begin{bmatrix} 1 & 1 & 1 \\ -\sqrt{\frac{3}{2}} & \sqrt{\frac{3}{2}} & 0 \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \sqrt{2} \end{bmatrix} . \begin{Bmatrix} \frac{2}{3} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{2}{3} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{2}{3} \end{Bmatrix} \begin{bmatrix} 1 & -\sqrt{\frac{3}{2}} & -\frac{1}{\sqrt{2}} \\ 1 & \sqrt{\frac{3}{2}} & -\frac{1}{\sqrt{2}} \\ 1 & 0 & \sqrt{2} \end{bmatrix}$$

$$= \frac{\beta}{3}\begin{bmatrix} 1 & 1 & 1 \\ -\sqrt{\frac{3}{2}} & \sqrt{\frac{3}{2}} & 0 \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \sqrt{2} \end{bmatrix} \begin{Bmatrix} \frac{1}{6} & -\frac{11}{12}\sqrt{\frac{3}{2}} & -\frac{11}{12}\frac{1}{\sqrt{2}} \\ \frac{1}{6} & \frac{11}{12}\sqrt{\frac{3}{2}} & -\frac{11}{12}\frac{1}{\sqrt{2}} \\ \frac{1}{6} & 0 & \frac{22}{12}\frac{1}{\sqrt{2}} \end{Bmatrix} \tag{9.74}$$

$$= \frac{\beta}{3}\begin{Bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{33}{12} & 0 \\ 0 & 0 & \frac{33}{12} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{6}\beta & 0 & 0 \\ 0 & \frac{11}{12}\beta & 0 \\ 0 & 0 & \frac{11}{12}\beta \end{Bmatrix},$$

this is the same result as was obtained in equation (9.57).

9.7 Moments Inertia for Different Body Coordinate Systems

We consider two sets of coordinate axes that are oriented in the same direction, but have different origins. The x_i-axes have their origin O located at the centre of mass of the rigid body, and the X_i-axes have their origin Q located somewhere else inside, or outside, of the body (see Figure 9-3).

The elements of the inertia tensor $\{\mathbf{J}\}$ relative to the X_i-axes are

$$J_{ij} = \sum_\alpha m_\alpha \left(\delta_{ij} X_{\alpha,k} X_{\alpha,k} - X_{\alpha,i} X_{\alpha,j}\right). \tag{9.75}$$

If the vector $\mathbf{a}$ connects the origin Q to the centre of mass (and origin) O, then the general vector $\mathbf{R}$ for the position of a point within the rigid body is written as $\mathbf{R}_\alpha = \mathbf{a} + \mathbf{r}_\alpha$, or using components

$$X_{\alpha,i} = a_i + x_{\alpha,i}. \tag{9.76}$$

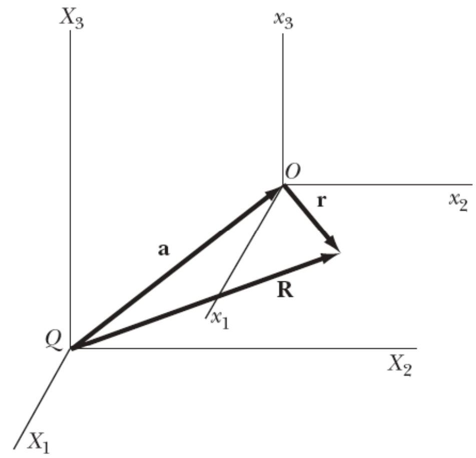

Figure 9-3 – The X_i-axes are fixed in the rigid body and have the same orientation as the x_i-axes, but its origin Q is not located at the same point O, which is the centre of mass of the body.

Inserting equation (9.76) into equation (9.75) we get

$$\begin{aligned} J_{ij} &= \sum_\alpha m_\alpha \left[\delta_{ij} \left(a_k + x_{\alpha,k} \right)\left(a_k + x_{\alpha,k} \right) - \left(a_i + x_{\alpha,i} \right)\left(a_j + x_{\alpha,j} \right) \right] \\ &= \sum_\alpha m_\alpha \left(\delta_{ij} x_{\alpha,k} x_{\alpha,k} - x_{\alpha,i} x_{\alpha,j} \right) \\ &\quad + \sum_\alpha m_\alpha \left[\delta_{ij} \left(2x_{\alpha,k} a_k + a_k a_k \right) - \left(a_i x_{\alpha,j} + a_j x_{\alpha,i} + a_i a_j \right) \right] \\ &= I_{ij} + \sum_\alpha m_\alpha \left(\delta_{ij} a_k a_k - a_i a_j \right) + \sum_\alpha m_\alpha \left(\delta_{ij} 2x_{\alpha,k} a_k - a_i x_{\alpha,j} - a_j x_{\alpha,i} \right). \end{aligned} \tag{9.77}$$

But from the definition of the centre of mass itself, the last term on the right hand side of the last of equations (9.77) equals zero since

$$\sum_\alpha m_\alpha x_{\alpha,i} = 0. \tag{9.78}$$

We then find the final result that

$$\boxed{J_{ij} = I_{ij} + M \left(\delta_{ij} a^2 - a_i a_j \right)} \tag{9.79}$$

with $M = \sum_\alpha m_\alpha$ and $a^2 = a_k a_k$.

We see from equation (9.79) that *the inertia tensor components are minimum when measured relative to the centre of mass*.

Example

Find the inertia tensor of a homogeneous cube of side b relative to its centre of mass.

Solution.
We previously found that the components of the inertia tensor of such a cube relative to one of its corner are given by

$$J_{11} = \frac{2}{3}\beta, \qquad J_{12} = -\frac{1}{4}\beta, \tag{9.80}$$

for the moments and the (negative of) the products of inertia, respectively. Since the centre of mass of the cube is located at $a_1 = a_2 = a_3 = b/2$ relative to a corner, we can use equation (9.79) to calculate the components I_{ij} of the inertia tensor relative to the centre of mass

$$\begin{aligned} I_{11} &= J_{11} - M\left(\frac{3}{4}b^2 - \frac{b^2}{4}\right) \\ &= \frac{2}{3}\beta - \frac{1}{2}\beta \\ &= \frac{1}{6}\beta, \end{aligned} \tag{9.81}$$

and

$$I_{12} = J_{12} + M\frac{b^2}{4} = 0. \tag{9.82}$$

The inertia tensor $\{\mathbf{I}\}$ is, therefore, seen to be diagonal and proportional to the unit tensor $\{\mathbf{1}\}$ with

$$\{\mathbf{I}\} = \frac{1}{6}Mb^2\{\mathbf{1}\}. \tag{9.83}$$

9.8 The Euler Angles

We stated in section 9.2 that of the six degrees of freedom of a rigid body, three are rotational in nature (the other three are for the translation motion of the centre of mass). In this section, we set to determine the set of angles that can be used to specify the rotation of a rigid body.

We know that the transformation from one coordinate system to another can be represented by a matrix equation such as

$$\mathbf{x} = \boldsymbol{\lambda}\mathbf{x}'. \tag{9.84}$$

If we identify the inertial (or fixed) system with $\mathbf{x}'$ and the rigid body coordinate system with $\mathbf{x}$, then the rotation matrix $\boldsymbol{\lambda}$ describes the relative orientation of the body in relation to the fixed system. Since there are three rotational degrees of freedom, $\boldsymbol{\lambda}$ is actually a product from three individual rotation matrices; one for each independent angles. Although there are many possible choices for the selection of these angles, we will use the so-called **Euler angles** ϕ, θ, and ψ.

The Euler angles are generated in the following series of rotation that takes the fixed $\mathbf{x}'$ system to the rigid body $\mathbf{x}$ system (see Figure 9-4).

1. The first rotation is counterclockwise through an angle ϕ about the x_3'-axis. It transforms the inertial system into an intermediate set of x_i''-axes. The transformation matrix is

$$\lambda_\phi = \begin{pmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{9.85}$$

with $0 \le \phi \le 2\pi$, and

$$\mathbf{x}'' = \boldsymbol{\lambda}_\phi \mathbf{x}'. \tag{9.86}$$

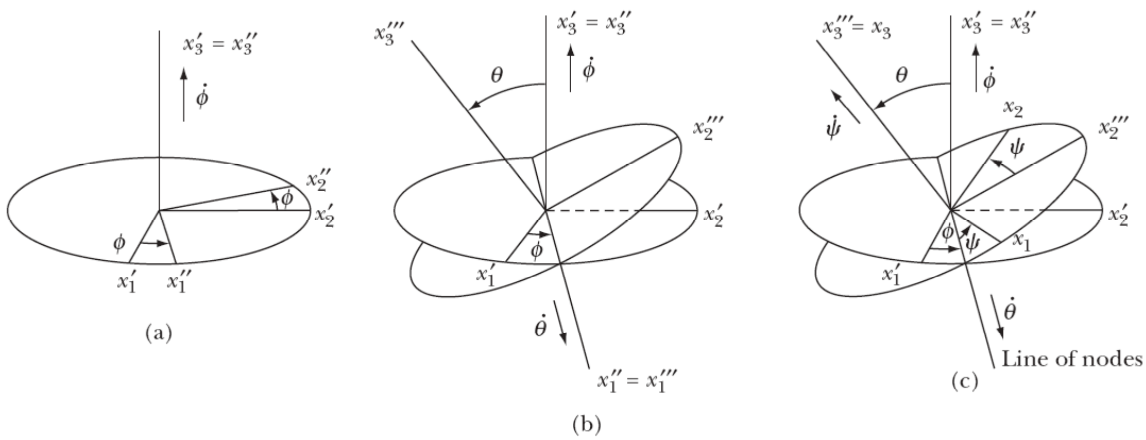

Figure 9-4 – The Euler angles are used to rotate the fixed $\mathbf{x}'$ system to the rigid body $\mathbf{x}$ system. (a) The first rotation is counterclockwise through an angle ϕ about the x_3'-axis. (b) The second rotation is counterclockwise through an angle θ about the x_1''-axis. (c) The third rotation is counterclockwise through an angle ψ about the x_3'''-axis.

2. The second rotation is counterclockwise through an angle θ about the x_1''-axis (also called the **line of nodes**). It transforms the inertial system into an intermediate set of x_i'''-axes. The transformation matrix is

$$\lambda_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{pmatrix}, \tag{9.87}$$

with $0 \leq \theta \leq \pi$, and

$$\mathbf{x}''' = \boldsymbol{\lambda}_\theta \mathbf{x}''. \tag{9.88}$$

3. The third rotation is counterclockwise through an angle ψ about the x_3'''-axis. It transforms the inertial system into the final set rigid body x_i-axes. The transformation matrix is

$$\lambda_\psi = \begin{pmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{9.89}$$

with $0 \leq \psi \leq 2\pi$, and

$$\mathbf{x} = \boldsymbol{\lambda}_\psi \mathbf{x}'''. \tag{9.90}$$

Combining the three rotations using equations (9.86), (9.88), and (9.90) we find that the complete transformation is given by

$$\mathbf{x} = \boldsymbol{\lambda}_\psi \boldsymbol{\lambda}_\theta \boldsymbol{\lambda}_\phi \mathbf{x}', \tag{9.91}$$

and the rotation matrix is

$$\boldsymbol{\lambda} = \boldsymbol{\lambda}_\psi \boldsymbol{\lambda}_\theta \boldsymbol{\lambda}_\phi. \tag{9.92}$$

Upon calculating this matrix, we find that its components are

$$
\begin{aligned}
\lambda_{11} &= \cos(\psi)\cos(\phi) - \cos(\theta)\sin(\phi)\sin(\psi) \\
\lambda_{21} &= -\sin(\psi)\cos(\phi) - \cos(\theta)\sin(\phi)\cos(\psi) \\
\lambda_{31} &= \sin(\theta)\sin(\phi) \\
\lambda_{12} &= \cos(\psi)\sin(\phi) + \cos(\theta)\cos(\phi)\sin(\psi) \\
\lambda_{22} &= -\sin(\psi)\sin(\phi) + \cos(\theta)\cos(\phi)\cos(\psi) \\
\lambda_{32} &= -\sin(\theta)\cos(\phi) \\
\lambda_{13} &= \sin(\psi)\sin(\theta) \\
\lambda_{23} &= \cos(\psi)\sin(\theta) \\
\lambda_{33} &= \cos(\theta).
\end{aligned} \tag{9.93}
$$

with $0 \le \phi \le 2\pi$, $0 \le \theta \le \pi$, and $0 \le \psi \le 2\pi$.

Correspondingly, the rate of change with time (i.e., the angular speed) associated with each of the three Euler angles are defined as $\dot{\theta}, \dot{\phi},$ and $\dot{\psi}$. The vectors associated with $\dot{\theta}, \dot{\phi},$ and $\dot{\psi}$ can be written as

$$
\begin{aligned}
\dot{\boldsymbol{\phi}} &= \dot{\phi}\mathbf{e}_3' \\
\dot{\boldsymbol{\theta}} &= \dot{\theta}\mathbf{e}_1'' \\
\dot{\boldsymbol{\psi}} &= \dot{\psi}\mathbf{e}_3''' = \dot{\psi}\mathbf{e}_3.
\end{aligned} \tag{9.94}
$$

Taking the projections of the unit bases vectors appearing in equation (9.94) on the rigid body bases vectors, we find

$$
\begin{aligned}
\dot{\boldsymbol{\phi}} &= \dot{\phi}\left[\sin(\theta)\sin(\psi)\mathbf{e}_1 + \sin(\theta)\cos(\psi)\mathbf{e}_2 + \cos(\theta)\mathbf{e}_3\right] \\
\dot{\boldsymbol{\theta}} &= \dot{\theta}\left[\cos(\psi)\mathbf{e}_1 - \sin(\psi)\mathbf{e}_2\right] \\
\dot{\boldsymbol{\psi}} &= \dot{\psi}\mathbf{e}_3.
\end{aligned} \tag{9.95}
$$

Combining the three equations (9.95), we can express the components of the total angular velocity vector $\boldsymbol{\omega}$ as a function of $\dot{\theta}, \dot{\phi},$ and $\dot{\psi}$

$$
\boxed{
\begin{aligned}
\omega_1 &= \dot{\phi}\sin(\theta)\sin(\psi) + \dot{\theta}\cos(\psi) \\
\omega_2 &= \dot{\phi}\sin(\theta)\cos(\psi) - \dot{\theta}\sin(\psi) \\
\omega_3 &= \dot{\phi}\cos(\theta) + \dot{\psi}
\end{aligned}
} \tag{9.96}
$$

9.9 Euler's Equations

To obtain the equations of motion of a rigid body, we can always start with the fundamental equation (see the dumbbell example on page 165)

$$\left(\frac{d\mathbf{L}}{dt}\right)_{\text{fixed}} = \mathbf{N}, \tag{9.97}$$

where $\mathbf{N}$ is the torque, and designation "fixed" is used since this equation can only applied in an inertial frame of reference. We also know from our study of noninertial frames of reference in Chapter 10 that

$$\left(\frac{d\mathbf{L}}{dt}\right)_{\text{fixed}} = \left(\frac{d\mathbf{L}}{dt}\right)_{\text{body}} + \boldsymbol{\omega}\times\mathbf{L}, \tag{9.98}$$

or

$$\left(\frac{d\mathbf{L}}{dt}\right)_{\text{body}} + \boldsymbol{\omega}\times\mathbf{L} = \mathbf{N}. \tag{9.99}$$

Using tensor notation we can write the components of equation (9.99) as

$$\dot{L}_i + \varepsilon_{ijk}\omega_j L_k = N_i. \tag{9.100}$$

Now, *if we chose the coordinate axes for the body frame of reference to coincide with the principal axes of the rigid body*, then we have from equations (9.44)

$$L_1 = I_1\omega_1, \qquad L_2 = I_2\omega_2, \qquad L_3 = I_3\omega_3. \tag{9.101}$$

Since the principal moments of inertia $I_1, I_2, \text{ and } I_3$ are constant with time, we can combine equations (9.100) and (9.101) to get

$$\begin{aligned}
I_1\dot{\omega}_1 - \left(I_2 - I_3\right)\omega_2\omega_3 &= N_1 \\
I_2\dot{\omega}_2 - \left(I_3 - I_1\right)\omega_3\omega_1 &= N_2 \\
I_3\dot{\omega}_3 - \left(I_1 - I_2\right)\omega_1\omega_2 &= N_3.
\end{aligned} \tag{9.102}$$

Alternatively, we can combine these three equations into one using indices

$$\boxed{\left(I_i - I_j\right)\omega_i\omega_j - \sum_k \varepsilon_{ijk}\left(I_k\dot{\omega}_k - N_k\right) = 0} \tag{9.103}$$

where *no summation* is implied on the i and j indices. Equations (9.103) are the so-called **Euler equations of motion** for a rigid body.

Examples

1. The dumbbell. We return to problem of the rotating dumbbell that we solved earlier (see page 165). Referring to equation (9.39) we found that the angular momentum was given by (using the rigid body coordinate system)

$$\mathbf{L} = \omega \sin(\theta)\left(m_1 r_1^{\,2} + m_2 r_2^{\,2}\right)\mathbf{e}_1, \tag{9.104}$$

with the principal moments of inertia from the system given by

$$\begin{aligned} I_1 &= I_2 = m_1 r_1^{\,2} + m_2 r_2^{\,2} \\ I_3 &= 0, \end{aligned} \tag{9.105}$$

and the angular velocity components by

$$\begin{aligned} \omega_1 &= \omega \sin(\theta) \\ \omega_2 &= 0 \\ \omega_3 &= \omega \cos(\theta). \end{aligned} \tag{9.106}$$

Since the system of axes chosen correspond to the principal axes of the dumbbell, then we can apply Euler's equations of motion (i.e., equations (9.102)). With the constraint that $\dot{\omega} = 0$, we find

$$\begin{aligned} N_1 &= 0 \\ N_2 &= I_1 \omega_3 \omega_1 = \left(m_1 r_1^{\,2} + m_2 r_2^{\,2}\right)\omega^2 \sin(\theta)\cos(\theta) \\ N_3 &= 0, \end{aligned} \tag{9.107}$$

or

$$\mathbf{N} = \left(m_1 r_1^{\,2} + m_2 r_2^{\,2}\right)\omega^2 \sin(\theta)\cos(\theta)\mathbf{e}_2. \tag{9.108}$$

Upon using equations (9.33) we can rewrite the torque in the inertial coordinate system (that shares a common origin with the body system) as

$$\mathbf{N} = \left(m_1 r_1^{\,2} + m_2 r_2^{\,2}\right)\omega^2 \sin(\theta)\cos(\theta)\left[-\sin(\omega t)\mathbf{e}_1' + \cos(\omega t)\mathbf{e}_2'\right]. \tag{9.109}$$

This is the same result as what was obtained with equations (9.42) without resorting to Euler's equation.

2. Force-free motion of a symmetric top.

We consider a symmetric top, whose principal moments of inertia are $I_1 = I_2 \neq I_3$, when no forces or torques are acting on it. In this case, the Euler's equations of motion are

$$\begin{aligned}
(I_1 - I_3)\omega_2\omega_3 - I_1\dot{\omega}_1 &= 0 \\
(I_3 - I_1)\omega_3\omega_1 - I_1\dot{\omega}_2 &= 0 \\
I_3\dot{\omega}_3 &= 0,
\end{aligned} \tag{9.110}$$

where I_1 (as opposed to I_2) was used throughout. Because there are no forces involved, the top will be either at rest or in uniform motion with respect to the inertial frame of reference. We will assume the top to be at rest, and located at the origin of the inertial coordinate axes. We want to find equations for the time evolution of the components of the angular velocity vector $\boldsymbol{\omega}$.

From the third of equations (9.110) we find that $\dot{\omega}_3 = 0$, or

$$\omega_3(t) = cste. \tag{9.111}$$

The equations for the other two components of the angular velocity yield

$$\begin{aligned}
\dot{\omega}_1 + \Omega\omega_2 &= 0 \\
\dot{\omega}_2 - \Omega\omega_1 &= 0,
\end{aligned} \tag{9.112}$$

with

$$\Omega = \left(\frac{I_3 - I_1}{I_1}\right)\omega_3. \tag{9.113}$$

Equations (9.112) form a system of coupled first order differential equations that can be solved in a similar fashion as was done for the problem of the Foucault pendulum in the previous chapter (see page 156). So, we multiply the second equation by i and add it to the first to get

$$\dot{\eta} - i\Omega\eta = 0, \tag{9.114}$$

with $\eta = \omega_1 + i\omega_2$. The solution to equation (9.114) is of the form

$$\eta(t) = Ae^{i\Omega t}, \tag{9.115}$$

or, alternatively,

$$\begin{aligned} \omega_1(t) &= A\cos(\Omega t) \\ \omega_2(t) &= A\sin(\Omega t). \end{aligned} \tag{9.116}$$

Since ω_3 is a constant, we find that the speed of rotation is also a constant. That is,

$$\omega^2 = \omega_1{}^2 + \omega_2{}^2 + \omega_3{}^2 = A^2 + \omega_3{}^2 = cste. \tag{9.117}$$

Because equations (9.116) are that of a circle, we find that the angular velocity vector $\boldsymbol{\omega}$ precesses about the top's axis of symmetry with a constant angular frequency Ω. To an observer attached to the rigid body coordinate system, $\boldsymbol{\omega}$ traces a cone around the x_3-axis, called the **body cone**. Also, since the top is not subjected to any forces or torques, the angular momentum $\mathbf{L}$ and the energy (which in this case reduces to the kinetic energy of rotation T_{rot}) are conserved. Therefore,

$$T_{\text{rot}} = \frac{1}{2}\mathbf{L}\cdot\boldsymbol{\omega} = cste, \tag{9.118}$$

with

$$\left(\frac{d\mathbf{L}}{dt}\right)_{\text{fixed}} = 0. \tag{9.119}$$

Equations (9.118) and (9.119) imply that the angular velocity vector $\boldsymbol{\omega}$ makes a constant angle with the (also constant) angular momentum vector $\mathbf{L}$. Therefore, not only does $\boldsymbol{\omega}$ precesses about the body x_3-axis, but it also precesses the axis that specifies the direction of $\mathbf{L}$. The angular velocity vector $\boldsymbol{\omega}$ traces a cone around the $\mathbf{L}$-axis, called the **space cone**.

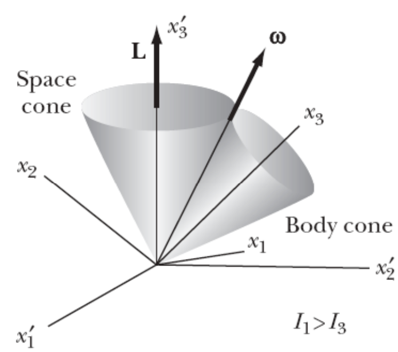

Figure 9-5 – The relative orientation of $\mathbf{L}$, $\boldsymbol{\omega}$, and the x_3-axis that result from the force-free motion of a symmetric top. We let $\mathbf{L}$ to lie along the x_3'-axis in the fixed coordinate system and $I_1 > I_3$. We can imagine the body cone rolling around the space cone.

We can find out the relative orientation of $\mathbf{L}$, $\boldsymbol{\omega}$, and the x_3-axis by calculating the following double product

$$\begin{aligned}\mathbf{L}\cdot(\boldsymbol{\omega}\times\mathbf{e}_3) &= L_i\varepsilon_{ijk}\omega_j\delta_{3k}\\ &= \varepsilon_{3ij}L_i\omega_j = L_1\omega_2 - L_2\omega_1\\ &= I_1\omega_1\omega_2 - I_2\omega_2\omega_1 = 0,\end{aligned} \tag{9.120}$$

since $I_1 = I_2$ for a symmetric top. We therefore have that $\mathbf{L}$, $\boldsymbol{\omega}$, and the x_3-axis all lie on the same plane. An example of this is shown in Figure 9-5 for the case where we let $\mathbf{L}$ to lie along the x_3'-axis in the fixed coordinate system and $I_1 > I_3$.

Finally, we can calculate the rate at which the rigid body will precess (about the x_3'-axis) in the inertial system. We know from the definition of the Euler angles that the rotation rate about the x_3'-axis is given by $\dot{\phi}$. Furthermore, the second Euler angle θ is that made between the x_3' (or $\mathbf{L}$) and the x_3 axes, with $\dot{\theta} = 0$ since this angle is constant. Then, using the first two of equations (9.96) we have

$$\omega_1^{\,2} + \omega_2^{\,2} = \dot{\phi}^2\sin^2(\theta). \tag{9.121}$$

If we define the components of the angular velocity vector $\boldsymbol{\omega}$ as a function of the angle α it makes to the x_3-axis, we have

$$\begin{aligned}\sqrt{\omega_1^{\,2} + \omega_2^{\,2}} &= \omega\sin(\alpha)\\ \omega_3 &= \omega\cos(\alpha).\end{aligned} \tag{9.122}$$

Since $I_1 = I_2$ we can also express the components of the angular momentum as

$$\begin{aligned}\sqrt{L_1^{\,2} + L_2^{\,2}} &= I_1\omega\sin(\alpha)\\ L_3 &= I_3\omega\cos(\alpha),\end{aligned} \tag{9.123}$$

or, alternatively,

$$\begin{aligned}\sqrt{L_1^{\,2} + L_2^{\,2}} &= L\sin(\theta)\\ L_3 &= L\cos(\theta).\end{aligned} \tag{9.124}$$

Combining equations (9.121) to (9.124) we find that

$$\dot{\phi} = \frac{\omega \sin(\alpha)}{\sin(\theta)} = \frac{L}{I_1}. \tag{9.125}$$

9.10 The Tennis Racket Theorem

The so-called tennis racket theorem is concerned with the stability of the rotational motion of a rigid body about its principal axis. We want to find out if, when a small perturbation is applied to the body, the motion either returns to its initial state or perform small oscillations about it (i.e., if it does not do this, then it is unstable).

We consider a general rigid body with principal moments of inertia such that $I_3 > I_2 > I_1$. We assume that the body coordinate axes are aligned with its principal axes, and first consider an initial rotation about the x_1-axis, and then apply small perturbations about the other two axes such that the angular velocity vector becomes

$$\boldsymbol{\omega} = \omega_1 \mathbf{e}_1 + \lambda \mathbf{e}_2 + \mu \mathbf{e}_3, \tag{9.126}$$

with $\lambda \ll \omega_1$ and $\mu \ll \omega_1$. Using equations (9.102) we can write the equations of motion for the system

$$\begin{aligned} (I_2 - I_3)\lambda\mu - I_1\dot{\omega}_1 &= 0 \\ (I_3 - I_1)\mu\omega_1 - I_2\dot{\lambda} &= 0 \\ (I_1 - I_2)\omega_1\lambda - I_3\dot{\mu} &= 0. \end{aligned} \tag{9.127}$$

If we will only keep terms of no higher than first order in λ and μ, the first of equations (9.127) imply that

$$\omega_1 = cste, \tag{9.128}$$

while the other two can be rewritten as

$$\begin{aligned} \dot{\lambda} &= \left(\frac{I_3 - I_1}{I_2}\omega_1\right)\mu \\ \dot{\mu} &= \left(\frac{I_1 - I_2}{I_3}\omega_1\right)\lambda, \end{aligned} \tag{9.129}$$

where the quantities in parentheses are constant. Taking the time derivative of the first of equations (9.129) and inserting the second in it we get

$$
\begin{aligned}
\ddot{\lambda} &= \left(\frac{I_3 - I_1}{I_2} \omega_1 \right) \dot{\mu} \\
&= \left[\frac{(I_3 - I_1)(I_1 - I_2)}{I_2 I_3} \omega_1^{\,2} \right] \lambda.
\end{aligned}
\tag{9.130}
$$

The solution to this second order differential equation is of the following form

$$
\lambda(t) = Ae^{i\Omega_1 t} + Be^{-i\Omega_1 t}, \tag{9.131}
$$

with

$$
\Omega_1 = \omega_1 \sqrt{\frac{(I_1 - I_3)(I_1 - I_2)}{I_2 I_3}}. \tag{9.132}
$$

Since $I_1 < I_3$ and $I_1 < I_2$, Ω_1 is real and the motion resulting from the perturbation is a bounded oscillation. Furthermore, upon inserting the result in equations (9.129) we find a similar result for $\mu(t)$. The rotation motion about the x_1-axis is therefore stable.

If we study next rotations about the x_2 and x_3-axis, we obtain similar results and identify the frequency of oscillations, stemming form the perturbations, by permutation of the indices in equation (9.132). That is,

$$
\begin{aligned}
\Omega_2 &= \omega_2 \sqrt{\frac{(I_2 - I_1)(I_2 - I_3)}{I_3 I_1}}. \\
\Omega_3 &= \omega_3 \sqrt{\frac{(I_3 - I_2)(I_3 - I_1)}{I_1 I_2}}.
\end{aligned}
\tag{9.133}
$$

But because $I_3 > I_2 > I_1$, we find that Ω_3 is also real, while Ω_2 is imaginary. Just as the motion about the x_1-axis was found to be stable, so is the motion about the x_3-axis. On the other hand, because Ω_2 is imaginary a perturbation will increase exponentially with time when the initial rotation is about the intermediate x_2-axis. Motion about this axis is thus unstable. We can readily test this result by spinning a rigid body (like a tennis racket!) about each of its principal axis.